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SUBSTRUCTURE MODEL UPDATING THROUGH ITERATIVE CONVEX OPTIMIZATION

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ABSTRACT

In order to obtain a more accurate finite element (FE) model for a built structure, experimental data collected from the actual structure can be used to update the FE model. This process is known as FE model updating. Numerous FE model updating algorithms have been developed in the past few decades. However, most existing algorithms suffer computational challenges, particularly when applied to a large structure with dense measurements. The reason is these approaches usually operate on a relatively complicated model for the entire structure. To address this issue, a substructure updating approach is presented in this paper. The Craig-Bampton theory is adopted to condense the entire structural model into a substructure (currently being analyzed) and a residual structure. Dynamic response of the residual structure is approximated using only a limited number of dominant mode shapes. To improve the convergence of this substructure approach for model updating, an iterative convex optimization procedure is developed and validated through numerical simulation with a 200 degrees-of-freedom spring-mass model. The proposed substructure model updating is shown to successfully detect the locations and severities of simulated damage.

INTRODUCTION

In modern structural analysis, a great amount of efforts have been devoted to developing accurate finite element (FE) models. However, predictions by numerical models often differ from experimental results. The discrepancy may be caused by various inaccuracies in numerical models. For example, in actual civil structures, member joints are far more complicated than frictionless hinges or fixed connections, although idealized joints are commonly used in FE models. Besides, most structural components may deteriorate over time. As a result, FE models based on the original structure cannot accurately describe the deteriorated structure. To obtain a more accurate model, experimental data collected from the actual structure in the field can be used to update the model, which is known as FE model updating. The updated model can predict structural response with higher fidelity. In addition, by tracking major property changes at individual structural components, model updating can assist in diagnosing structural damage.

Numerous FE model updating algorithms have been developed in the past few decades [1]. Among these algorithms, one major category is modal-based approaches. These approaches update system parameters by forming an optimization problem that minimizes the difference in modal parameters between experimental measurements and FE simulations. Early researchers started model updating by minimizing difference between measured and simulated natural frequencies. For example, Zhang *et al*. proposed an eigenvalue sensitivity-based model updating approach and applied it on a scaled suspension bridge model [2]. Salawu reviewed various model updating algorithms using natural frequencies, and concluded that changes in frequencies may not be sufficient enough for identifying system parameters [3]. Therefore, other modal characteristics, e.g. mode shapes or modal flexibility, were investigated for model updating. For example, Moller and Friberg adopted the modal assurance criterion (MAC)-related function for updating the model of an industrial structure [4]. FE model updating using changes in mode shapes and frequencies was applied for damage assessment of a reinforced concrete beam [5]. More recently, Yuen developed an efficient model updating algorithm using frequencies and mode shapes at only some selected degrees of freedom (DOFs) for a few modes [6]. Jaishi and Ren proposed an objective function consisting of changes in frequency, MAC related function and modal flexibility for updating the model of a beam structure [7]. Nevertheless, previous approaches generally suffer computational difficulties while updating the model of a large-scale structure with dense measurements, because the approaches usually operate on the entire structural model that can have a large number of DOFs.

In order to address the computational difficulty, particularly to accommodate data collected at dense measurement locations, substructure-based FE model updating has been investigated. A well-known substructure modeling method is the Craig-Bampton theory that partitions a large structure into a substructure been analyzed and a residual structure containing the rest of DOFs [8, 9]. Dynamic response of the residual structure is approximated using only a limited number of dominant mode shapes, so that the large structural model is condensed to a simplified model with much small number of DOFs. Such a sub/residual-structure approach for FE model updating was studied in [10], using a laboratory 2D rectangular frame with free boundary conditions.

Besides suffering computational difficulty, previous optimization objective functions that describe the difference in experimental and simulated modal parameters are highly nonlinear and non-convex to updating parameters. As a result, conventional modal-based approaches suffer convergence issues, and may not guarantee global optimum. To overcome the difficulty, this research investigates efficient substructure model updating through the formulation of a convex optimization problem. The convex attribute guarantees global optimality of an optimization problem, and makes the solution process tractable and highly efficient [11, 12]. Moreover, some DOFs are difficult to measure during field experiments, such as rotational DOFs. To obtain experimental mode shapes for all DOFs, a modal expansion process is adopted to obtain the complete mode shapes using measurement at a limited number of DOFs. Partly due to modal expansion, formulation of the convex optimization problem is based upon an initial FE model. Therefore, an iterative convex optimization procedure is proposed for higher model updating accuracy. After an updated model is obtained as the solution of convex optimization, the updated model is used again as an initial model to repeat the updating process till the solution converges. The rest of the paper is organized as follows. The substructure approach based on Craig-Bampton theory is presented first, followed by the formulation of iterative convex optimization procedure for substructure model updating. The proposed algorithm is then validated through numerical simulation of a 200-DOF spring-mass model. Finally, a summary and discussion are provided.

SUBSTRUCTURE MODELING

Fig. 1 illustrates substructure modeling following Craig-Bampton theory [8, 9]. Subscripts s, i, and r are used to denote the substructure being analyzed, the interface nodes, and the residual structure, respectively. The block-bidiagonal structural

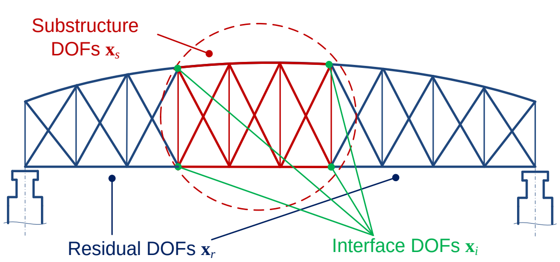

Fig. 1 Illustration of substructure modeling.

stiffness and mass matrices, $\mathbf{K}$ and $\mathbf{M}$, can be assembled using original DOFs $\mathbf{x} = \begin{bmatrix} \mathbf{x}_s & \mathbf{x}_i & \mathbf{x}_r \end{bmatrix}^{\mathrm{T}}$:

$$\begin{aligned}\mathbf{K} &= \begin{bmatrix} \begin{bmatrix} & \mathbf{K}_S & \end{bmatrix} & \begin{matrix}\mathbf{0}\\ \mathbf{0}\end{matrix} \\ \begin{matrix}\mathbf{0} & \mathbf{0}\end{matrix} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \begin{matrix}\mathbf{0} & \mathbf{0}\end{matrix} \\ \begin{matrix}\mathbf{0}\\ \mathbf{0}\end{matrix} & \begin{bmatrix} & \mathbf{K}_R & \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{si} \\ \mathbf{K}_{is} & \mathbf{K}_{ii}^S \end{bmatrix} & \begin{matrix}\mathbf{0}\\ \mathbf{0}\end{matrix} \\ \begin{matrix}\mathbf{0} & \mathbf{0}\end{matrix} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \begin{matrix}\mathbf{0} & \mathbf{0}\end{matrix} \\ \begin{matrix}\mathbf{0}\\ \mathbf{0}\end{matrix} & \begin{bmatrix} \mathbf{K}_{ii}^R & \mathbf{K}_{ir} \\ \mathbf{K}_{ri} & \mathbf{K}_{rr} \end{bmatrix} \end{bmatrix}\end{aligned} \tag{1}$$

$$\begin{aligned}\mathbf{M} &= \begin{bmatrix} \begin{bmatrix} & \mathbf{M}_S & \end{bmatrix} & \begin{matrix}\mathbf{0}\\ \mathbf{0}\end{matrix} \\ \begin{matrix}\mathbf{0} & \mathbf{0}\end{matrix} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \begin{matrix}\mathbf{0} & \mathbf{0}\end{matrix} \\ \begin{matrix}\mathbf{0}\\ \mathbf{0}\end{matrix} & \begin{bmatrix} & \mathbf{M}_R & \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{si} \\ \mathbf{M}_{is} & \mathbf{M}_{ii}^S \end{bmatrix} & \begin{matrix}\mathbf{0}\\ \mathbf{0}\end{matrix} \\ \begin{matrix}\mathbf{0} & \mathbf{0}\end{matrix} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \begin{matrix}\mathbf{0} & \mathbf{0}\end{matrix} \\ \begin{matrix}\mathbf{0}\\ \mathbf{0}\end{matrix} & \begin{bmatrix} \mathbf{M}_{ii}^R & \mathbf{M}_{ir} \\ \mathbf{M}_{ri} & \mathbf{M}_{rr} \end{bmatrix} \end{bmatrix}\end{aligned} \tag{2}$$

where $\mathbf{K}_S$ and $\mathbf{M}_S$ denote the stiffness and mass matrices of the substructure (free interface); $\mathbf{K}_R$ and $\mathbf{M}_R$ denote the stiffness and mass matrices of the residual structure (free interface); $\mathbf{K}_{ii}^S$ and $\mathbf{M}_{ii}^S$ denote the stiffness and mass entries for the interface DOFs and contributed by the substructure; $\mathbf{K}_{ii}^R$ and $\mathbf{M}_{ii}^R$ denote the stiffness and mass entries for the interface DOFs and contributed by the residual structure.

The dynamic behavior of the residual structure can be approximated using Craig-Bampton formulation [8, 9]. The DOFs of the residual structure, $\mathbf{x}_r \in \mathbb{R}^{n_r}$, are approximated by a linear combination of interface DOFs, $\mathbf{x}_i \in \mathbb{R}^{n_i}$, and modal coordinates of the residual structure, $\mathbf{q}_r \in \mathbb{R}^{n_q}$.

$$\mathbf{x}_r \approx \mathbf{T}\mathbf{x}_i + \mathbf{\Phi}_r \mathbf{q}_r \tag{3}$$

where $\mathbf{T} = -\mathbf{K}_{rr}^{-1}\mathbf{K}_{ri}$ is the Guyan static condensation matrix. $\mathbf{\Phi}_r = \begin{bmatrix} \boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_{n_q} \end{bmatrix}$ represents the mode shapes of the residual structure fixed at the interface.

$$\left(-\omega_r^{\ 2}\mathbf{M}_{rr} + \mathbf{K}_{rr}\right)\boldsymbol{\varphi}_r = 0 \tag{4}$$

Although the size of the residual structure may be large, the number of modal coordinates, n_q, can be chosen as relatively small to reflect the first few dominant mode shapes only (i.e. $n_q << n_r$). The transformation matrix $\mathbf{\Gamma}$ can be formulated with reduced dimension $(n_i+n_r) \times (n_i+n_q)$:

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{I} & \\ \mathbf{T} & \mathbf{\Phi}_r \end{bmatrix} \tag{5}$$

such that

$$\begin{bmatrix} \mathbf{x}_i \\ \mathbf{x}_r \end{bmatrix} \approx \mathbf{\Gamma} \begin{bmatrix} \mathbf{x}_i \\ \mathbf{q}_r \end{bmatrix} \tag{6}$$

Suppose $\tilde{\mathbf{K}}_R$ and $\tilde{\mathbf{M}}_R$ denote the new stiffness and mass matrices of the residual structure after transformation:

$$\tilde{\mathbf{K}}_R = \mathbf{\Gamma}^{\mathrm{T}} \mathbf{K}_R \mathbf{\Gamma} = \begin{bmatrix} \tilde{\mathbf{K}}_{ii} & \\ & \boldsymbol{\gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{ii}^R + \mathbf{T}^{\mathrm{T}} \mathbf{K}_{ri} & \\ & \mathbf{\Phi}_r^{\mathrm{T}} \mathbf{K}_{rr} \mathbf{\Phi}_r \end{bmatrix} \tag{7}$$

$$\begin{aligned} \tilde{\mathbf{M}}_R &= \mathbf{\Gamma}^{\mathrm{T}} \mathbf{M}_R \mathbf{\Gamma} = \begin{bmatrix} \tilde{\mathbf{M}}_{ii} & \tilde{\mathbf{M}}_{iq} \\ \tilde{\mathbf{M}}_{qi} & \boldsymbol{\mu} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{M}_{ii}^R + \mathbf{T}^{\mathrm{T}} \mathbf{M}_{ri} + \mathbf{M}_{ir} \mathbf{T} + \mathbf{T}^{\mathrm{T}} \mathbf{M}_{rr} \mathbf{T} & (\mathbf{M}_{ir} + \mathbf{T}^{\mathrm{T}} \mathbf{M}_{rr}) \mathbf{\Phi}_r \\ \mathbf{\Phi}_r^{\mathrm{T}} (\mathbf{M}_{ri} + \mathbf{M}_{rr} \mathbf{T}) & \mathbf{\Phi}_r^{\mathrm{T}} \mathbf{M}_{rr} \mathbf{\Phi}_r \end{bmatrix} \end{aligned} \tag{8}$$

where $\boldsymbol{\gamma} = \mathrm{diag}(\gamma_1, \cdots, \gamma_{n_q})$ and $\boldsymbol{\mu} = \mathrm{diag}(\mu_1, \cdots, \mu_{n_q})$ are diagonal modal stiffness and modal mass matrices of the residual structure fixed at the interface. Note that due to the static condensation process in this transformation, the off-diagonal block components of $\tilde{\mathbf{K}}_R$ are zero.

Under transformation to the residual structure, a new set of stiffness matrix $\tilde{\mathbf{K}}$ and structural mass matrix $\tilde{\mathbf{M}}$ of the entire structure can be assembled, while contribution from the substructure, $\mathbf{K}_S$ and $\mathbf{M}_S$ (Eq. (1)), remains unchanged.

$$\begin{aligned} \tilde{\mathbf{K}} &= \begin{bmatrix} \begin{bmatrix} \mathbf{K}_S \end{bmatrix} & \begin{matrix} \mathbf{0} \\ \mathbf{0} \end{matrix} \\ \begin{matrix} \mathbf{0} & \mathbf{0} \end{matrix} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \begin{matrix} \mathbf{0} & \mathbf{0} \end{matrix} \\ \begin{matrix} \mathbf{0} \\ \mathbf{0} \end{matrix} & \begin{bmatrix} \tilde{\mathbf{K}}_R \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{si} & \mathbf{0} \\ \mathbf{K}_{is} & \mathbf{K}_{ii}^S + \tilde{\mathbf{K}}_{ii} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\gamma} \end{bmatrix} \end{aligned} \tag{9}$$

$$\begin{aligned} \tilde{\mathbf{M}} &= \begin{bmatrix} \begin{bmatrix} \mathbf{M}_S \end{bmatrix} & \begin{matrix} \mathbf{0} \\ \mathbf{0} \end{matrix} \\ \begin{matrix} \mathbf{0} & \mathbf{0} \end{matrix} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \begin{matrix} \mathbf{0} & \mathbf{0} \end{matrix} \\ \begin{matrix} \mathbf{0} \\ \mathbf{0} \end{matrix} & \begin{bmatrix} \tilde{\mathbf{M}}_R \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{M}_{ss} & \mathbf{M}_{si} & \mathbf{0} \\ \mathbf{M}_{is} & \mathbf{M}_{ii}^S + \tilde{\mathbf{M}}_{ii} & \tilde{\mathbf{M}}_{iq} \\ \mathbf{0} & \tilde{\mathbf{M}}_{qi} & \boldsymbol{\mu} \end{bmatrix} \end{aligned} \tag{10}$$

Link proposed a model updating approach for matrices of both the substructure and the residual structure [10], where the substructure model is updated as

$$\mathbf{K}_S = \mathbf{K}_{S0} + \sum_{m=1}^{n_\alpha} \alpha_m \mathbf{K}_{S0,m} \qquad \mathbf{M}_S = \mathbf{M}_{S0} + \sum_{n=1}^{n_\beta} \beta_n \mathbf{M}_{S0,n} \tag{11}$$

where $\mathbf{K}_{S0}$ and $\mathbf{M}_{S0}$ are the stiffness and mass matrices of the substructure and used as initial starting point in the model updating; α_m and β_n correspond to physical system parameters to be updated, such as elastic modulus and density of each element; n_α and n_β represent the total number of updating system parameters; $\mathbf{K}_{S0,m}$ and $\mathbf{M}_{S0,n}$ are constant matrices determined by the type and location of these physical parameters. For the rest of this paper, subscript "0" will be used to denote variables associated with the initial structural model, which serves as the starting point for model updating.

Assuming that physical parameter changes in the residual structure do not alter the mode shapes significantly, the transformed residual structural model is updated by

$$\tilde{\mathbf{K}}_R = \tilde{\mathbf{K}}_{R0} + \sum_{j=1}^{n_q+n_i} p_j \tilde{\mathbf{K}}_{R0,j} \qquad \tilde{\mathbf{M}}_R = \tilde{\mathbf{M}}_{R0} + \sum_{j=1}^{n_q+n_i} q_j \tilde{\mathbf{M}}_{R0,j} \tag{12}$$

where p_j and q_j are variables to be updated; $\tilde{\mathbf{K}}_{R0}$ and $\tilde{\mathbf{M}}_{R0}$ are the initial stiffness and mass matrices of the transformed residual structure model; $\tilde{\mathbf{K}}_{R0,j}$ and $\tilde{\mathbf{M}}_{R0,j}$ represent the constant correction matrices formulated using modal back-transform:

$$\tilde{\mathbf{K}}_{R0,j} = \boldsymbol{\varphi}_{R0,j}^l \omega_{R0,j}^2 \boldsymbol{\varphi}_{R0,j}^{l,\mathrm{T}} \qquad \tilde{\mathbf{M}}_{R0,j} = \boldsymbol{\varphi}_{R0,j}^l \boldsymbol{\varphi}_{R0,j}^{l,\mathrm{T}} \tag{13}$$

where

$$\begin{aligned} \mathbf{\Phi}_{R0}^l &= \mathbf{\Phi}_{R0}^{-\mathrm{T}} = \begin{bmatrix} \boldsymbol{\varphi}_{R0,1} & \cdots & \boldsymbol{\varphi}_{R0,n_i+n_q} \end{bmatrix}^{-\mathrm{T}} \\ &= \begin{bmatrix} \boldsymbol{\varphi}_{R0,1}^l & \cdots & \boldsymbol{\varphi}_{R0,n_i+n_q}^l \end{bmatrix} \end{aligned} \tag{14}$$

$\boldsymbol{\varphi}_{R0,j}$ and $\omega_{R0,j}$ are the *j*-th normalized mode shape and the *j*-th resonance frequency of the initial transformed residual structural model with free interface:

$$\mathbf{\Phi}_{R0}^{\mathrm{T}} \tilde{\mathbf{M}}_{R0} \mathbf{\Phi}_{R0} = \mathbf{I} \qquad \mathbf{\Phi}_{R0}^{\mathrm{T}} \tilde{\mathbf{K}}_{R0} \mathbf{\Phi}_{R0} = \mathrm{diag}(\omega_{R0,1}^2, \cdots, \omega_{R0,(n_q+n_i)}^2) \tag{15}$$

Using the model matrices to be updated, i.e. Eq. (11) for substructure and Eq. (12) for residual structure, the entire structural model with reduced DOFs $\begin{bmatrix} \mathbf{x}_s & \mathbf{x}_i & \mathbf{q}_r \end{bmatrix}^{\mathrm{T}}$ can be updated with variables α_m, β_n, p_j, and q_j. For brevity, these variables will be referred to in their vector form as $\boldsymbol{\alpha} \in \mathbb{R}^{n_\alpha}$, $\boldsymbol{\beta} \in \mathbb{R}^{n_\beta}$, $\mathbf{p} \in \mathbb{R}^{n_q+n_i}$ and $\mathbf{q} \in \mathbb{R}^{n_q+n_i}$.

$$\tilde{\mathbf{K}} = \tilde{\mathbf{K}}_0 + \sum_{m=1}^{n_\alpha} \alpha_m \begin{bmatrix} \begin{bmatrix} \mathbf{K}_{S0,m} \end{bmatrix} & \mathbf{0} \\ & \mathbf{0} \\ \mathbf{0} \quad \mathbf{0} & \mathbf{0} \end{bmatrix} + \sum_{j=1}^{n_i+n_q} p_j \begin{bmatrix} \mathbf{0} & \mathbf{0} \quad \mathbf{0} \\ \mathbf{0} & \\ \mathbf{0} & \begin{bmatrix} \tilde{\mathbf{K}}_{R0,j} \end{bmatrix} \end{bmatrix} \tag{16}$$

$$\tilde{\mathbf{M}} = \tilde{\mathbf{M}}_0 + \sum_{n=1}^{n_\beta} \beta_n \begin{bmatrix} \begin{bmatrix} \mathbf{M}_{S0,n} \end{bmatrix} & \mathbf{0} \\ & \mathbf{0} \\ \mathbf{0} \quad \mathbf{0} & \mathbf{0} \end{bmatrix} + \sum_{j=1}^{n_i+n_q} q_j \begin{bmatrix} \mathbf{0} & \mathbf{0} \quad \mathbf{0} \\ \mathbf{0} & \\ \mathbf{0} & \begin{bmatrix} \tilde{\mathbf{M}}_{R0,j} \end{bmatrix} \end{bmatrix} \tag{17}$$

ITERATIVE CONVEX OPTIMIZATION

As discussed in the introduction, previous model updating approaches usually formulate an objective function using the difference between experimental and simulated frequencies and mode shapes. However, such formulations usually suffer convergence difficulty, and cannot guarantee global optimum. To address this issue, convex optimization can be adopted for model updating [13]. The convex attribute guarantees global optimality of the solution, and makes the solution process tractable and highly efficient [11, 12].

To give the definition of convex optimization, the concepts of convex set and convex function are first described. A set $\mathbb{C} \subset \mathbb{R}^n$ is convex if the line segment between any two points in $\mathbb{C}$ lies in $\mathbb{C}$. In other words, a set $\mathbb{C}$ is convex if for any $\mathbf{x}$ and $\mathbf{y} \in \mathbb{C}$ and any θ with $0 \le \theta \le 1$, we have

$$\theta\mathbf{x} + (1-\theta)\mathbf{y} \in \mathbb{C} \tag{18}$$

A function f: $\mathbb{R}^n \to \mathbb{R}$ is convex if **dom** f (domain of f) is a convex set, and if for all $\mathbf{x}$, $\mathbf{y} \in$ **dom** f and any θ with $0 \le \theta \le 1$, we have

$$f\left(\theta\mathbf{x} + (1-\theta)\mathbf{y}\right) \le \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}) \tag{19}$$

In other words, for a convex function, the function value at any linear interpolation between any two points $\mathbf{x}$ and $\mathbf{y}$ must be smaller than or equal to the linear interpolation of the function values at points $\mathbf{x}$ and $\mathbf{y}$. Note that for a scalar affine function[1] f: $\mathbb{R}^n \to \mathbb{R}$, the equality in Eq. (19) always holds. Therefore, a scalar affine function is also convex.

Finally, a convex optimization problem has following form:

$$\begin{aligned} &\text{Minimize} && f_0(\mathbf{x}) && && \text{(20a)} \\ &\text{Subject to} && f_i(\mathbf{x}) \le 0, && i = 1, \ldots, m && \text{(20b)} \\ & && \mathbf{a}_i^{\mathrm{T}}\mathbf{x} = b_i\,, && i = 1, \ldots, n && \text{(20c)} \end{aligned}$$

[1] In general, a function f: $\mathbb{R}^n \to \mathbb{R}^m$ is affine if it is a sum of a linear function and a constant, i.e., if it has the form: $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

where $f_0, \ldots, f_m$ are convex functions; $\mathbf{a}_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ are constant. For an optimization problem as defined in Eq. (20), the feasible set of optimization variables $\mathbf{x} \in \mathbb{R}^n$, which satisfy constraints in Eq. (20b, c), is a convex set.

In this study, a convex optimization formulation is proposed for substructure model updating with optimization variables $\mathbf{x} = [\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{p}, \mathbf{q}, \mathbf{s}]^{\mathrm{T}}$, where $\boldsymbol{\alpha} \in \mathbb{R}^{n_\alpha}$, $\boldsymbol{\beta} \in \mathbb{R}^{n_\beta}$, $\mathbf{p} \in \mathbb{R}^{n_q+n_i}$, $\mathbf{q} \in \mathbb{R}^{n_q+n_i}$, and $\mathbf{s} \in \mathbb{R}^m$:

$$\begin{aligned} &\text{Minimize} && \max\left(s_1, s_2, \cdots, s_m\right) && \text{(21a)} \\ &\text{Subject to} && && \\ & && \left\| \left(-\omega_j^2 \tilde{\mathbf{M}}(\boldsymbol{\beta}, \mathbf{q}) + \tilde{\mathbf{K}}(\boldsymbol{\alpha}, \mathbf{p})\right) \boldsymbol{\psi}_j \right\| - s_j \le 0; \quad j = 1 \ldots m && \text{(21b)} \\ & && \boldsymbol{\alpha}_L \le \boldsymbol{\alpha} \le \boldsymbol{\alpha}_U \qquad \boldsymbol{\beta}_L \le \boldsymbol{\beta} \le \boldsymbol{\beta}_U && \\ & && \mathbf{p}_L \le \mathbf{p} \le \mathbf{p}_U \qquad \mathbf{q}_L \le \mathbf{q} \le \mathbf{q}_U && \text{(21c)} \end{aligned}$$

where m denotes the number of measured modes obtained from experimental data; $\mathbf{s} = [s_1, s_2, \cdots, s_m]^{\mathrm{T}}$ includes slack variables limiting quadratic eigenvalue expressions in Eq. (21b); $\boldsymbol{\alpha}_L$, $\boldsymbol{\beta}_L$, $\mathbf{p}_L$ and $\mathbf{q}_L$ denote the lower bounds for vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{p}$, and $\mathbf{q}$, respectively; $\boldsymbol{\alpha}_U$, $\boldsymbol{\beta}_U$, $\mathbf{p}_U$ and $\mathbf{q}_U$ denote the upper bounds for vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{p}$, and $\mathbf{q}$, respectively; ω_j and $\boldsymbol{\psi}_j$ represent the j-th frequency and mode shape of the transformed structural model (Eq. (9)), obtained from experimental data. Note that the sign "≤" in Eq. (21c) is overloaded to represent element-wise inequality.

The objective function (21a) is convex because max function is a convex function [11]. This satisfies the requirement that $f_0(\mathbf{x})$ in Eq. (20a) is convex. The matrices $\tilde{\mathbf{K}}(\boldsymbol{\alpha}, \mathbf{p})$ and $\tilde{\mathbf{M}}(\boldsymbol{\beta}, \mathbf{q})$ are affine functions on optimization variables $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{p}$, and $\mathbf{q}$ (Eq. (16) and (17)). The eigen-pairs ω_j and $\boldsymbol{\psi}_j$, obtained from experimental data, are constant in the optimization problem. Therefore, the quadratic eigenvalue expression in the inequality constraint, $\left(-\omega_j^2 \tilde{\mathbf{M}}(\boldsymbol{\beta}, \mathbf{q}) + \tilde{\mathbf{K}}(\boldsymbol{\alpha}, \mathbf{p})\right) \boldsymbol{\psi}_j$, is still affine on variables $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{p}$, and $\mathbf{q}$. The composition of a norm function and an affine function remains convex, so $\left\| \left(-\omega_j^2 \tilde{\mathbf{M}}(\boldsymbol{\beta}, \mathbf{q}) + \tilde{\mathbf{K}}(\boldsymbol{\alpha}, \mathbf{p})\right) \boldsymbol{\psi}_j \right\|$ remains a convex function. In addition, $-s_j$ is affine on $\mathbf{s}$, and thus a convex function on $\mathbf{s}$. Because a nonnegative weighted sum of convex functions remains convex, the function at the left hand side of Eq. (21b) is also convex. Therefore, Eq. (21b) is equivalent to Eq. (20b) with a convex function $f_i(\mathbf{x})$ on optimization variables $\mathbf{x} = [\boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{p}, \mathbf{q}, \mathbf{s}]^{\mathrm{T}}$. Besides, associated inequalities in the lower and upper bound constraints (Eq. (21c)) can be readily rewritten into the form that is also equivalent to Eq. (20b) with an appropriate affine (and thus convex) function $f_i(\mathbf{x})$. Therefore, the proposed optimization formulation (Eq. (21)) satisfies the definition of a convex optimization problem (Eq. (20)). Note that a convex optimization problem does not necessary have to possess either the equality constraints (Eq. (20c)) or the inequality constraints (Eq. (20b)).

Some DOFs are difficult to measure during experiment, such as rotational DOFs. Besides, the mode shapes corresponding to the generalized coordinates q_r cannot be physically measured. Therefore, to obtain experimental mode shapes for the transformed structural model ($\tilde{\mathbf{K}}$, $\tilde{\mathbf{M}}$), a modal expansion process can be performed [14]:

$$\boldsymbol{\psi}_{j,U} = (-\mathbf{D}_{UU}^{-1}\mathbf{D}_{UM})\boldsymbol{\psi}_{j,M} \tag{22}$$

where subscripts M and U represent the measured and unmeasured DOFs, respectively; $\boldsymbol{\psi}_{j,M}$ and $\boldsymbol{\psi}_{j,U}$ represent the measured and unmeasured parts of the j-th mode shape vector. The expansion matrix $(-\mathbf{D}_{UU}^{-1}\mathbf{D}_{UM})$ comes from the eigen-problem of the transformed initial structural model (Eq. (9) and (10)):

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_{MM} & \mathbf{D}_{MU} \\ \mathbf{D}_{UM} & \mathbf{D}_{UU} \end{bmatrix} = -\omega_j^2\tilde{\mathbf{M}}_0 + \tilde{\mathbf{K}}_0 \tag{23}$$

As described before, the model updating process is based upon an initial FE model. This is reflected by $\tilde{\mathbf{K}}_0$ in Eq.(16), $\tilde{\mathbf{M}}_0$ in Eq. (17), and both in Eq. (23). The updated model, as solution to the optimization problem in Eq. (21), can be used as an initial model again to repeat the updating process for higher accuracy. The procedures of the iterative convex optimization process for updating one substructure are summarized as:

1. Formulate the initial FE model ($\tilde{\mathbf{K}}_0$, $\tilde{\mathbf{M}}_0$);
2. Modal expansion using experimental data and the initial FE model (Eq. (22));
3. Solve convex optimization problem for **α**, **β**, **p**, **q** and **s** (Eq. (21));
4. Use updated **α**, **β**, **p** and **q** to update $\tilde{\mathbf{K}}_0$ and $\tilde{\mathbf{M}}_0$ and return to step 2, until the updated variables converge.

After updating the current substructure, the substructure model updating process can be applied to other substructures until the entire structural model is updated.

NUMERICAL EXAMPLE

To validate the proposed convex optimization approach for substructure model updating, simulation is performed with a 200-DOF spring-mass model (Fig. 2). For the initial model, all the mass and spring stiffness values are set identically as 6kg and 35kN/m, respectively. Damage is introduced to this model by reducing 10% of spring stiffness to k_{20}, k_{30}, k_{45}, k_{50}, k_{60}, k_{62}, k_{82}, k_{100}, k_{120}, and k_{150}. Fig. 3 shows a conceptual drawing of the 200-DOF spring-mass numerical model with marked damage

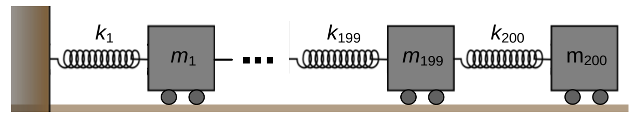

Fig. 2 A multi-DOF spring-mass system

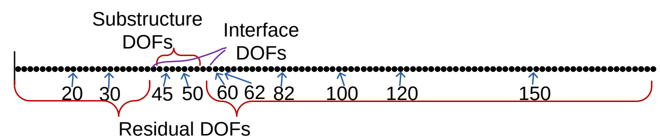

Fig. 3 Illustration of damage locations and substructure selection

locations. A substructure with DOFs from 41 to 54 is selected for model updating. DOFs 40 and 55 are interface DOFs, and all other DOFs belong to the residual structure.

Dynamic response of the residual structure is approximated using six modal coordinates, i.e. $n_q = 6$ (Eq. (3)). The first six mode shapes of the residual structure fixed at the interface, $\boldsymbol{\Phi}_r$ (Eq. (3)), are plotted in the manner for an equivalent fictional 200-story shear building (Fig. 4). With $\mathbf{x}_s \in \mathbb{R}^{14\times1}$ and $\mathbf{x}_i \in \mathbb{R}^{2\times1}$, the entire structural model is condensed to 22 DOFs. Note that two springs with stiffness loss, k_{45} and k_{50}, are contained in the substructure. Assuming acceleration measurements are available only on the substructure and interface DOFs, the objective is to identify the damage using proposed substructure updating approach.

Updating parameters are selected as α_1, α_2, ..., α_{15} and p_1, p_2, ..., p_7 (Eq. (16)). Parameters α_1, α_2, ..., and α_{15} denote relative stiffness changes in k_{41}, k_{42}, ..., and k_{55} that belong to the substructure, respectively; p_1, p_2, ..., and p_7 denote the modal space parameters of the residual structure with free interface.

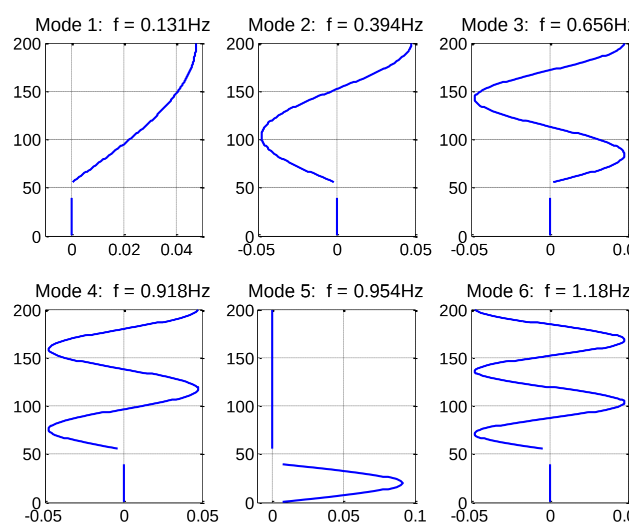

Fig. 4 First six mode shapes of the residual structure fixed at the interface

Note that although in general n_i+n_q number of modal parameters need to be updated (Eq. (12)), only seven updating modal parameters are necessary in this example. The reason is that because the first resonance frequency of the residual structure with free interface is zero (corresponding to free-body movement), the first modal correction matrix $\tilde{\mathbf{K}}_{R0,1}$(Eq. (13)) is zero. It is assumed that we have perfect knowledge in mass, so $\boldsymbol{\beta}$ and $\mathbf{q}$ are not the among the optimization variables.

Modal characteristics of the damaged structure are calculated from the mass and stiffness matrices of the damaged structure. For simplicity, the first three natural frequencies and mode shapes on all substructure and interface DOFs (DOFs 40 to 55) are directly used as the experimentally measured eigen-pairs. Fig. 5 shows the first three mode shapes of the entire damaged structure, where the mode shapes on DOFs 40 to 55 are highlighted. The mode shapes for the six modal coordinates $\mathbf{q}_r$ are expanded using Eq. (22). The model updating process starts from the initial undamaged model. The upper and lower bounds in Eq. (21c) are set to 20% and -20% for each updating variable.

The iterative convex optimization process is conducted. The set of optimal parameters obtained after ten iterations are shown in Table I, Table II and Table III. Table I shows the optimal values for α_5 and α_{10} are -10.1% and -10.1%, respectively, which indicates that the stiffness of k_{45} and k_{50} are reduced by 10.1% and 10.1%. The optimal values for all other α_i are very close to zero, which implies small variations in all other spring stiffnesses in the substructure. The results match the damage locations and severities. Table II illustrates an optimal set of non-zero modal parameters. Their changes are due to stiffness loss in the residual structure. Table III shows the optimal slack variables for the inequality constrains in Eq. (21b). The slack variables are close to zero, which indicates that the updated model has modal characteristics that are very close to the damaged structure. Fig. 6 presents the convergence plot for some representative stiffness ratios. It can be concluded that the proposed algorithm offers fast convergence speed, i.e. the stiffness ratios obtained after the first iteration are already closed to the expected results.

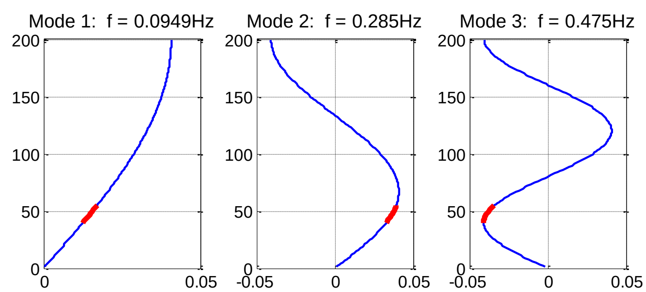

Fig. 5 First three modes of the damaged structure

Table I. Optimal stiffness changes for substructure elements

Stiffness changes	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8
Optimal value (%)	-0.1	-0.1	-0.1	-0.1	-10.1	-0.1	-0.1	-0.1
Stiffness changes	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	
Optimal value (%)	-0.1	-10.1	-0.1	-0.1	-0.1	-0.1	-0.1	

Table II. Optimal modal parameters

Modal parameters	p_1	p_2	p_3	p_4	p_5	p_6	p_7
Optimal value (%)	-0.4	-1.2	-0.7	-0.8	-0.8	-0.8	4.0

Table III. Optimal slack variables (Eq. (21b))

Slack variables	s_1	s_2	s_3
Optimal value	0.035	0.035	0.035

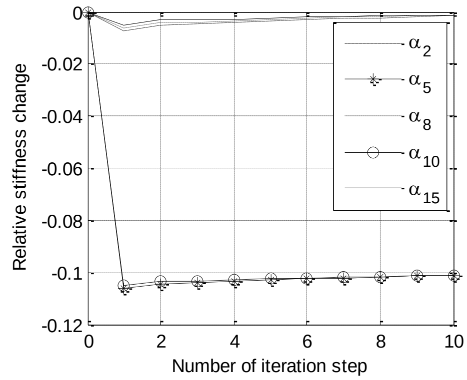

Fig. 6 Convergence of stiffness ratios

CONCLUSIONS

In this paper, a substructure model updating approach is proposed. The Craig-Bampton theory is adopted to simplify a large structure into a substructure currently being analyzed and a residual structure. Dynamic response of the residual structure is approximated using only a limited number of dominant mode shapes. To improve the convergence for model updating, an iterative convex optimization formulation is proposed and validated through numerical simulation of a 200-DOF spring-mass model. The proposed substructure model updating is shown to successfully detect the damage locations and severities introduced in the numerical model. Future research will continue to investigate the proposed algorithm on more complicated structural models through numerical simulations and laboratory experiments.

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